Composite Young diagrams, supercharacters of U(M/N) and modification rules

1987 J. Phys. A: Math. Gen. 20 3121

(http://iopscience.iop.org/0305-4470/20/11/018)

J. Phys. A: Math. Gen. **20** (1987) 3121-3133. Printed in the UK

Composite Young diagrams, supercharacters of U(*M*/*N*) and modification rules

C J Cummins and R C King

Mathematics Department, University of Southampton, Southampton SO9 5NH, UK

Received 24 September 1986

Abstract. The supercharacter of $U(M/N)$ associated with an arbitrary composite Young diagram is defined. The distinction is made between standard and non-standard supercharacters. A modification rule is presented which may be used to express any non-standard supercharacter in terms of standard supercharacters. In the case $N=0$ the rule reduces to the rule already known to be appropriate to $U(M)$.

1. Introduction

In recent years composite Young diagrams have been introduced (Dondi and Jarvis 1981, Balantekin and Bars 1981a, b, Bars *et al* 1983, Bars 1984, King 1983a, b) into the study of representations of $U(M/N)$ and correspondingly of the Lie superalgebra $A(M-1, N-1)$. Such composite Young diagrams, although not essential to the study of $U(M)$ and the corresponding Lie algebra A_{M-1}, have been exploited in this context to considerable effect (King 1970, Abramsky and King 1970, King 1971, Black *et al* 1983). In the domain of $U(M/N)$ where composite Young diagrams cannot be avoided by conversion to ordinary Young diagrams it might be expected that they should play an important role.

Unfortunately, however, there exists a complication arising from the fact that Lie superalgebras possess atypical irreducible representations (Kac 1978). The work of Delduc and Gourdin (1984, 1985) makes it clear that the relationship between the supercharacters associated with composite Young diagrams and irreducible supercharacters is complicated and certainly not one-to-one in the atypical case. In this paper, composite Young diagrams are used to define certain supercharacters of $U(M/N)$ which can be thought of as supersymmetric functions of a set of indeterminates (King 1983b). For composite Young diagrams corresponding to typical irreducible representations these supersymmetric functions may be shown to coincide, under the identification of the indeterminates with formal exponentials, with the irreducible typical supercharacters defined by Kac (1978) for $A(M-1, N-1)$. Other composite Young diagrams correspond to atypical irreducible representations but the associated supercharacters now coincide with a linear combination of irreducible atypical supercharacters.

It should be pointed out that although we choose to work with supercharacters rather than characters in deference to the fact that supercharacters are associated with the fully $U(M/N)$ invariant supertrace operation (Balantekin and Bars 1981a), it is only necessary to change certain sign factors to pass from supercharacters to characters and vice versa. These sign changes will be referred to where appropriate in the text. It suffices to say that one advantage of the Young diagram approach adopted here is that these sign factors are easy to relate to Young diagram parameters. Moreover, the tensor product formula, a key determinantal expansion and the modification rule with which this paper is concerned are all valid for both characters and supercharacters.

As stressed by Delduc and Gourdin (1984, 1985) only a subset of all composite Young diagrams is needed to specify all irreducible tensor representations—typical and atypical—of U(M/N). The members of this subset they called legal. Here the word 'standard' is preferred and is applied where appropriate to both composite Young diagrams and the corresponding supercharacters. Non-standard supercharacters are not necessarily zero and when they arise in the consideration of tensor products or branching rules they cannot be simply ignored or set aside. Instead it is essential to express such non-standard supercharacters in terms of standard supercharacters by means of modification rules analogous to those already introduced for U(M), O(M) and Sp(M) (King 1971, Black *et al* 1983) and, more recently, for OSp(M/N) (Cummins and King 1987). With this in mind, the properties of supercharacters of U(M/N) defined in terms of U(M)$\times$U(N) characters are explored by taking the values of M and N to be arbitrarily large and then using the known modification rules for U(M) and U(N) to derive the required modification rules for U(M/N).

In the next section a reminder is given of the definition of composite Young diagrams and the modification rule for U(M) (King 1971) is stated. This is followed by a statement of the U(M/N) modification rule which it is our intention to prove and it is then made clear that this rule carries with it the definition of standardness for U(M/N) supercharacters.

The supercharacters of U(M/N) associated with Young diagrams are defined explicitly in § 3 and the validity of a very important determinantal expansion of supercharacters (Balantekin and Bars 1981b) is established. This is used in § 4 in the derivation of the modification rule, which is followed in § 5 by some illustrations of the application of the modification rule to both typical and atypical non-standard supercharacters of U(M/N).

2. Composite Young diagrams and modification rules

The composite Young diagram $F(\bar{\tau};\sigma)$, specified by the pair of partitions $\sigma=(\sigma_1,\sigma_2,\ldots)$ and $\tau=(\tau_1,\tau_2,\ldots)$, consists of two conventional Young diagrams $F(\sigma)$ and $F(\tau)$. The former is composed of boxes arranged in left-adjusted rows of lengths $\sigma_1,\sigma_2,\ldots$, and the latter of dotted boxes arranged in right-adjusted rows of lengths $\tau_1,\tau_2,\ldots$. The manner of juxtaposition of $F(\sigma)$ and $F(\tau)$ to form $F(\bar{\tau};\sigma)$ is to some extent a matter of taste but here the original back-to-back notation (Abramsky and King 1970, King 1970) is refined by reflecting the dotted part of the diagram in the horizontal line at the top of the diagram. By way of illustration, for $(\bar{\tau};\sigma)=(\overline{3^2 21};2^2 1^4)$ the composite Young diagram is displayed in figure 1.

The character of the irreducible mixed tensor representation of U(M) corresponding to $F(\bar{\tau};\sigma)$ is conveniently denoted by $\{\bar{\tau};\sigma\}$. Such characters of U(M) are standard if and only if the number of parts σ_1' and τ_1' of the partitions σ and τ, respectively, are such that their sum is less than or equal to M. It should be noted that this is precisely the condition that $F(\bar{\tau};\sigma)$ fits inside a horizontal strip of depth M. Characters not satisfying this condition are said to be non-standard. They are related to standard characters via the U(M) modification rule

$$\{\bar{\tau};\sigma\}=(-1)^{c+\bar{c}+1}\{\overline{\tau-h};\sigma-h\} \qquad \text{if } h=\sigma_1'+\tau_1'-M-1\geqslant 0 \qquad (2.1)$$

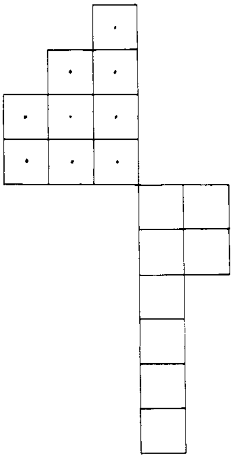


Figure 1.

where $\sigma - h$ and $\tau - h$ specify diagrams $F(\sigma - h)$ and $F(\tau - h)$ obtained from $F(\sigma)$ and $F(\tau)$, respectively, by the removal of continuous boundary strips of boxes each of length h starting at the foot of the first columns of $F(\sigma)$ and $F(\tau)$ and extending over c and $\bar{c}$ columns respectively. The boundary strip is said to be removable if $h \geq 0$ and if the juxtaposition of $F(\sigma - h)$ and $F(\tau - h)$ yields a regular composite Young diagram of the form $F(\bar{\beta}; \alpha)$. In such a case $\{\overline{\tau - h}; \sigma - h\}$ is to be interpreted as $\{\bar{\beta}; \alpha\}$. If $F(\sigma - h)$ is irregular in that no partition α exists such that $F(\sigma - h) = F(\alpha)$ or $F(\tau - h)$ is irregular in that no partition β exists such that $F(\tau - h) = F(\beta)$ then the strip h is not removable and $\{\overline{\tau - h}; \sigma - h\}$ is to be interpreted as being identically zero.

In the case of U(3) and $\{\bar{\tau}; \sigma\} = \{\overline{3^2 21}; 2^2 1^4\}$, for example, it can be seen first that $h = 6$ and then from figure 2 that $c = 2$ and $\bar{c} = 3$ leading to the identity $\{\overline{3^2 21}; 2^2 1^4\} = \{\overline{21}; 2\}$.

If necessary the modification rule (2.1) should be applied more than once until either $h < 0$ or until the character in question is shown to be zero. This leads immediately

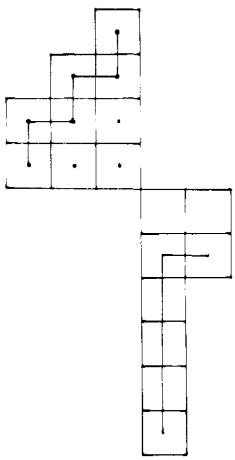


Figure 2.

to the U(M) standardness condition

$$\sigma_1' + \tau_1' \leqslant M. \tag{2.2}$$

In the case of U(M/N) a similar notation and terminology may be employed. Supercharacters associated with mixed tensor representations of U(M/N) are again denoted by $\{\bar{\tau}; \sigma\}$. Such a supercharacter and the associated composite Young diagram are said to be U(M/N) standard if and only if there exists at least one pair (j, l) such that (Delduc and Gourdin 1984, 1985)

$$\sigma_j' + \tau_l' \leqslant M \qquad \text{with } j + l = N + 2 \tag{2.3}$$

where σ_j' and τ_l' are the lengths of the jth and lth columns of $F(\sigma)$ and $F(\tau)$, respectively. The U(M/N) modification rule for non-standard supercharacters then takes the form

$$\{\bar{\tau}; \sigma\} = \sum_{k=1}^{N+1} \sum_{(j,l)} (-1)^{c+1} \{\overline{\tau - h_{j_1 l_1} - h_{j_2 l_2} - \ldots - h_{j_k l_k}}; \sigma - h_{j_1 l_1} - h_{j_2 l_2} - \ldots - h_{j_k l_k}\} \tag{2.4}$$

where

$$c = c_{j_1} + \bar{c}_{l_1} + c_{j_2} + \bar{c}_{l_2} + \ldots + c_{j_k} + \bar{c}_{l_k} \tag{2.5}$$

and

$$h_{jl} = \sigma_j' - j + \tau_l' - l - M + N + 1 \geqslant 0. \tag{2.6}$$

The second summation is over all pairs (j, l) of sequences $(j) = (j_1, j_2, \ldots, j_k)$ and $(l) = (l_1, l_2, \ldots, l_k)$ such that $N+1 \geqslant j_1 > j_2 > \ldots > j_k \geqslant 1$, $1 \leqslant l_1 < l_2 < \ldots < l_k \leqslant N+1$ and

$$j_i + l_i = N + 2 \qquad \text{for } i = 1, 2, \ldots, k. \tag{2.7}$$

The notation is such that $\sigma - h_{jl}$ specifies a diagram obtained from $F(\sigma)$ by the removal of a continuous boundary strip of length h_{jl} starting at the foot of the jth column and extending over c_j columns, whilst $\tau - h$ specifies a diagram obtained from $F(\tau)$ by the removal of a continuous boundary strip starting at the foot of the lth column and extending over $\bar{c}_l$ columns. The order in which these strips are removed from $F(\sigma)$ is given by the sequence (j) and the order in which they are removed from $F(\tau)$ is given by the reverse of the sequence (l). By virtue of the constraint (2.7) the strip length given by (2.5) simplifies to

$$h_{jl} = \sigma_j' + \tau_l' - M - 1 \geqslant 0. \tag{2.8}$$

Some examples will be given later which illustrate the use of the modification rule (2.4). For the moment it is to be noted only that this rule implies the validity of the standardness condition (2.3), which in turn implies that $F(\bar{\tau}; \sigma)$ and the corresponding supercharacter $\{\bar{\tau}; \sigma\}$ are standard if and only if $F(\bar{\tau}; \sigma)$ fits inside a cross, the horizontal portion of which has depth M and the vertical portion of which has width N (King 1986) as shown in figure 3.

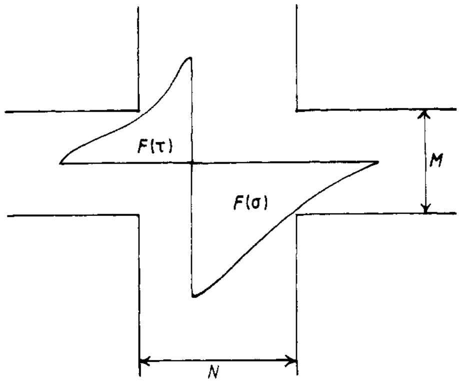


Figure 3.

3. Supercharacters of U(*M*/*N*)

It is of course necessary to give a precise definition of the supercharacters $\{\bar{\tau};\sigma\}$ of U(M/N) referred to in the previous section. This can most readily be done by means of the formal branching rule from U(M/N) to U(M)×U(N):

$$\{\bar{\tau};\sigma\}=\sum_{\alpha,\beta,\gamma}(-1)^{|\beta|-|\alpha|}\{\overline{\tau/\beta};\sigma/\alpha\}\{\overline{\beta'/\gamma};\alpha'/\gamma\} \tag{3.1}$$

where α and β are partitions of $|\alpha|$ and $|\beta|$, and α' and β' denote their conjugates. For supercharacters that correspond to irreducible representations of U(M/N) this formula yields the U(M)×U(N) content of the representation. In the general case where $\{\bar{\tau};\sigma\}$ is a linear combination of supercharacters of irreducible representations of U(M/N) (3.1) expresses this linear combination in terms of U(M)×U(N) characters. Since the characters of U(M) and U(N) are themselves symmetric functions, (3.1) serves to define the supercharacter $\{\bar{\tau};\sigma\}$ of U(M/N) as what might be called a supersymmetric function. The formula (King 1983b) for the corresponding character, as opposed to the supercharacter, of U(M/N), is obtained from (3.1) merely by the omission of the sign factor $(-1)^{|\beta|-|\alpha|}$.

As a consequence of the definition (3.1) it is not difficult to show, by comparing products of characters of U($M+N$) expressed in terms of characters of U(M)×U(N) with products of supercharacters of U(M/N) expressed via (3.1) in the same way, that

$$\{\bar{\pi};\mu\}\times\{\bar{\tau};\sigma\}=\sum_{\alpha,\beta}\{\overline{(\pi/\beta)(\tau/\alpha)};(\mu/\alpha)(\sigma/\beta)\}. \tag{3.2}$$

This product rule, it should be stressed, applies both to characters and supercharacters of U(M/N) and is also identical to that which applies to U(M). The only difference arises from the proper interpretation of any non-standard terms which may arise on the right-hand side. In the case of U(M) (2.1) must be used to make this interpretation, whilst for U(M/N) characters and supercharacters it is the main purpose of this paper to prove that (2.4) is the required modification rule.

In order to carry out this proof it is first necessary to justify the validity of our major tool—a determinantal expansion of the supercharacter $\{\bar{\tau};\sigma\}$ due to Balantekin and Bars (1981b).

The character $\{\mu\}$ of an irreducible covariant tensor representation of U(M) is nothing other than the symmetric function known as a Schur or S function. It is well known (Littlewood 1940, p 89) that

$$\{\mu\} = |\{1^{\mu'_j - j + i}\}|. \tag{3.3}$$

The row and column indices i and j range over the values $1, 2, \ldots, m$ with $m \geqslant \mu_1$ where

$$\{1^t\} = 0 \qquad \text{for } t < 0. \tag{3.4}$$

The fundamental modification rule for $\mathrm{U}(M)$ linking the characters of covariant and contravariant representations takes the form

$$\{\bar{1}^{M-t}\} = \bar{\varepsilon}\{1^t\} \qquad \text{for all } t \tag{3.5}$$

where ε denotes the one-dimensional alternating or determinant character of $\mathrm{U}(M)$ and $\bar{\varepsilon} = \varepsilon^{-1}$. Clearly (3.4) and (3.5) together imply that

$$\{1^t\} = \{\bar{1}^t\} = 0 \qquad \text{for both } t < 0 \text{ and } t > M. \tag{3.6}$$

Of course (3.5) is just an example of a more general equivalence relation between characters of covariant tensor and mixed tensor representations of $\mathrm{U}(M)$. Indeed quite generally (King 1970)

$$\{\bar{\tau}; \sigma\} = \varepsilon^{\tau_1}\{\mu\} \tag{3.7}$$

where $F(\mu)$ is obtained from $F(\bar{\tau}; \sigma)$ by taking the complement with respect to M of the dotted boxes in the first τ_1 columns of $F(\bar{\tau}; \sigma)$ and lowering them by reflection in their base line to give τ_1 columns of undotted boxes adjacent to those of $F(\sigma)$ and thereby forming the Young diagram $F(\mu)$ (rows not left-adjusted in this case). This key result implies firstly that it is not really necessary to use composite Young diagrams in the $\mathrm{U}(M)$ context, but secondly if they are to be used then the corresponding characters may be defined by (3.7). This definition is appropriate whether or not $\{\bar{\tau}; \sigma\}$ is standard. Indeed in the non-standard case the modification rule (2.1) for $\mathrm{U}(M)$ was first derived (King 1971) by making use of (3.7).

Taking the definition (3.7), using the determinantal expansion (3.3) and applying (3.5) to all the entries in the first τ_1 columns of this determinant then yields the identity postulated by Balantekin and Bars (1981b):

$$\{\bar{\tau}; \sigma\} = \begin{vmatrix} \{\bar{1}^{\tau'_T}\} & \cdots & \{\bar{1}^{\tau'_2+T-2}\} & \{\bar{1}^{\tau'_1+T-1}\} & \{1^{\sigma'_1-T}\} & \{1^{\sigma'_2-T-1}\} & \cdots & \{1^{\sigma'_s-S-T+1}\} \\ \{\bar{1}^{\tau'_T-1}\} & \cdots & \{\bar{1}^{\tau'_2+T-3}\} & \{\bar{1}^{\tau'_1+T-2}\} & \{1^{\sigma'_1-T+1}\} & \{1^{\sigma'_2-T}\} & \cdots & \{1^{\sigma'_s-S-T+2}\} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \{\bar{1}^{\tau'_T-T+1}\} & \cdots & \{\bar{1}^{\tau'_2-1}\} & \{\bar{1}^{\tau'_1}\} & \{1^{\sigma'_1-1}\} & \{1^{\sigma'_2-2}\} & \cdots & \{1^{\sigma'_s-S}\} \\ \{\bar{1}^{\tau'_T-T}\} & \cdots & \{\bar{1}^{\tau'_2-2}\} & \{\bar{1}^{\tau'_1-1}\} & \{1^{\sigma'_1}\} & \{1^{\sigma'_2-1}\} & \cdots & \{1^{\sigma'_s-S+1}\} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \{\bar{1}^{\tau'_T-S-T+2}\} & \cdots & \{\bar{1}^{\tau'_2-S}\} & \{\bar{1}^{\tau'_1-S+1}\} & \{1^{\sigma'_1+S-2}\} & \{1^{\sigma'_2+S-3}\} & \cdots & \{1^{\sigma'_s-1}\} \\ \{\bar{1}^{\tau'_T-S-T+1}\} & \cdots & \{\bar{1}^{\tau'_2-S-1}\} & \{\bar{1}^{\tau'_1-S}\} & \{1^{\sigma'_1+S-1}\} & \{1^{\sigma'_2+S-2}\} & \cdots & \{1^{\sigma'_s}\} \end{vmatrix} \tag{3.8}$$

where $S \geqslant \sigma_1$ and $T \geqslant \tau_1$, and this is an $m \times m$ determinant with $m = S + T$. For example

$$\{\bar{1}^2; 2^3\} = \begin{vmatrix} \{\bar{1}^2\} & \{1^2\} & \{1\} \\ \{\bar{1}\} & \{1^3\} & \{1^2\} \\ \{\bar{0}\} & \{1^4\} & \{1^3\} \end{vmatrix}. \tag{3.9}$$

Having derived this key formula (3.8) for $\mathrm{U}(M)$ it is easy to see that it must also apply in the case of $\mathrm{U}(M/N)$ since the expansion of the determinant yields a sum of products which are to be evaluated using (3.2) whether it is $\mathrm{U}(M)$ or $\mathrm{U}(M/N)$ under consideration. No modification rules are needed to ensure that the right-hand side reduces identically to the single term of the left-hand side. The formula (3.8) is, needless to say, valid for both characters and supercharacters.

Just as in the case of OSp(M/N) (Cummins and King 1987), it is the absence in the case of U(M/N) of a totally antisymmetric tensor analogous to $\varepsilon_{i_1 i_2 \ldots i_M}$ of U(M) which prohibits the validity of a simple rule like (3.5) for U(M/N). Nonetheless it is possible to define the supercharacter ε corresponding to the superdeterminant of U(M/N) by means of the formula

$$\varepsilon = \{1^M\} \times \{\bar{1}^N\}. \qquad (3.10)$$

This will be employed in the next section which is concerned with the derivation of the modification rule (2.4).

4. Derivation of modification rule

As in the derivation of the modification rule for OSp(M/N) (Cummins and King 1987) it is now convenient to introduce certain matrices whose role will only subsequently become clear. First of all column matrices C^p and $\bar{C}^r$ are introduced whose ath elements are given in terms of U(M/N) supercharacters by

$$C_a^p = \{1^{p+a-1}\} \qquad \bar{C}_a^r = \{\bar{1}^{r-a+1}\} \qquad (4.1)$$

where $1 \leq a \leq m$ and p and r are arbitrary integers. From the definition (3.1) it then follows that, in terms of U(M)×U(N) characters,

$$C_a^p = \sum_{t=0}^{\infty} (-1)^{p-t+i-1}\{1^t\} \times \{p-t+i-1\} \qquad (4.2)$$

and

$$\bar{C}_a^r = \sum_{t=0}^{\infty} (-1)^{r-t-i+1}\{\bar{1}^t\} \times \{\overline{r-t-i+1}\} \qquad (4.3)$$

where account has been taken of (3.6).

Next define the row matrix R_q whose bth element is given in terms of characters of U(M)×U(N) by

$$R_q^b = \{0\} \times \{1^{q-b}\} \qquad (4.4)$$

where $1 \leq b \leq m$ and $1 \leq q \leq m$. It then follows that the product matrices $K = RC$ and $\bar{K} = R\bar{C}$ have matrix elements given by

$$K_q^p = \sum_{t=0}^{\infty} (-1)^{p-t}\{1^t\} \times \{p-t, 1^{q-1}\} \qquad (4.5)$$

and

$$\bar{K}_q^r = \sum_{t=0}^{\infty} (-1)^{r-t}\{\bar{1}^t\} \times \{\overline{r-t}; 1^{q-1}\}. \qquad (4.6)$$

Now taking the modification rule (2.1) into account

$$K_q^p = \{1^{p+q-1}\} \times \{0\} \qquad (4.7)$$

and

$$\bar{K}_q^r = \{\bar{1}^{N+p-q+1}\} \times \{1^N\} \qquad (4.8)$$

for $N+1 \leqslant q \leqslant m$. Hence it can be seen that

$$K^p_q = \varepsilon \bar{K}_q^{M-N-p} \qquad \text{and} \qquad \bar{K}^r_q = \bar{\varepsilon} K_q^{M-N-r} \qquad \text{for } N+1 \leqslant q \leqslant m. \tag{4.9}$$

It should be noted that the $m \times m$ determinantal expression (3.8) for the supercharacter $\{\bar{\tau}; \sigma\}$ has a column structure such that

$$\{\bar{\tau}; \sigma\} = |\theta_{b,T} \bar{C}^{r_b}_a + \theta_{T+1,b} C^{p_b}_a| \tag{4.10}$$

where

$$p_b = \sigma'_{b-T} - b + 1 \qquad r_b = \tau'_{T-b+1} + b - 1 \tag{4.11}$$

and

$$\theta_{a,b} = \begin{cases} 1 & \text{if } 1 \leqslant a \leqslant b \\ 0 & \text{if } b < a. \end{cases} \tag{4.12}$$

The row and column indices a and b range over the values $1, 2, \ldots, m$ where

$$m = S + T \qquad \text{with } S = \sigma_1 \text{ and } T = \tau_1. \tag{4.13}$$

In fact the parameter S may be chosen to be any integer greater than or equal to σ_1, and similarly T may be chosen to be any integer greater than or equal to τ_1. In what follows it is essential to constrain S and T so that

$$S \geqslant \max(\sigma_1, N+1) \qquad T \geqslant \max(\tau_1, N+1). \tag{4.14}$$

With this in mind the next step is the introduction of a matrix D whose matrix elements are given by

$$\begin{aligned} D_{bc} = {} & \theta_{c,T} \bar{C}^{r_c}_b + \theta_{T+1,c} C^{p_c}_b + (\theta_{T-N,c} - \theta_{T+1,c}) \bar{x}_{c-T+N+1} \varepsilon \bar{C}_b^{M-N-p'_c} \\ & + (\theta_{T+1,c} - \theta_{T+N+2,c}) x_{c-T} \bar{\varepsilon} C_b^{M-N-r'_c} \end{aligned} \tag{4.15}$$

where

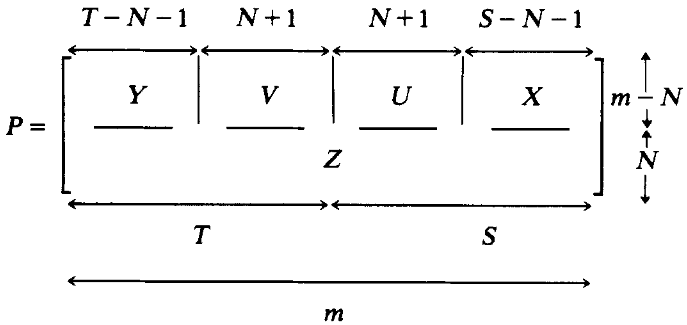
$$p'_c = p_{c+N+1} \qquad r'_c = r_{c-N-1} \tag{4.16}$$

whilst x_i for $i = 1, 2, \ldots, N+1$ are a set of indeterminates with inverses denoted by $\bar{x}_i$.

The final magic ingredient is the very special $m \times m$ matrix M with matrix elements

$$M_{ab} = \theta_{a,m-N} R^b_{m-a+1} + \theta_{m-N+1,a} \delta^b_{m-a+1}. \tag{4.17}$$

Multiplication of D by M yields an $m \times m$ matrix $P = MD$ which may be naturally partitioned into five submatrices as shown in the following diagram:



(4.18)

The $(m-N) \times (N+1)$ submatrices U and V have matrix elements given by

$$U_{ij} = P_{i,j+T} = K^{p''_j}_{m-i+1} + x_j \bar{\varepsilon} K^{M-N-r''_j}_{m-i+1} \tag{4.19a}$$

and

$$V_{ij} = P_{i,j+T-N-1} = \bar{K}^{r_j''}_{m-i+1} + \bar{x}_j \varepsilon \bar{K}^{M-N-p_j''}_{m-i+1} \tag{4.19b}$$

for $1 \leqslant i \leqslant m-N$ and $1 \leqslant j \leqslant N+1$, where

$$p_j'' = p_{j+T} \qquad r_j'' = r_{j+T-N-1}. \tag{4.20}$$

Now at last the reward may be reaped for all the previous manipulations. The application of (4.9) to (4.19) gives

$$U_{ij} = x_j V_{ij} \qquad \text{for } 1 \leqslant i \leqslant m-N, 1 \leqslant j \leqslant N+1. \tag{4.21}$$

It follows further that the $m \times m$ matrix P possesses $N+1$ pairs of columns with $m-N$ elements of one column given by the multiple x_j of the corresponding elements of the other column of each pair. Therefore subtracting these multiples of columns from their partners gives an $(m-N) \times (N+1)$ block of zeros, and hence

$$\det P = 0. \tag{4.22}$$

However $P = MD$ with

$$\det M = (-1)^{(m-N)(m+N-1)/2} \tag{4.23}$$

as can be seen from (4.7). Thus

$$\det D = 0. \tag{4.24}$$

Moreover as can be seen from (4.15) $\det D$ is a polynomial in the indeterminates x_j and their inverses $\bar{x}_j$. The expansion of $\det D$ in terms of these indeterminates yields a sum of 2^{2N+2} determinants. Since $\det D = 0$ for all values of the x_j the constant term in this expansion independent of all the x_j must itself be zero. This term, consisting of a sum of 2^{N+1} determinants, then gives the identity

$$\sum_{\boldsymbol{\eta}} |\theta_{c,T} \bar{C}_b^{r_c - \eta_c h_c} + \theta_{T+1,c} C_b^{p_c - \eta_c h_c}| = 0 \tag{4.25}$$

where

$$h_c = p_c + r_c' - M + N = p_c' + r_c - M + N \tag{4.26}$$

and the summation is carried out over all vectors $\boldsymbol{\eta}$ such that the components η_c are 0 or 1, with η_c only allowed to take the value 1 for pairs

$$c = T + j \qquad c = T - l + 1 \qquad \text{with } j + l = N + 2. \tag{4.27}$$

For these values of c it follows from (4.11), (4.16) and (4.26) that

$$h_c = h_{jl} = \sigma_j' - j + \tau_l' - l - M + N + 1 \tag{4.28}$$

as in (2.5).

The required modification rule (2.4) then follows immediately given the usual connection (King 1971) between the removal of continuous boundary strips and the regularisation of Young diagrams by the successive transposition of columns.

5. Illustration and discussion

At first sight the $\mathrm{U}(M/N)$ modification rule (2.4) may seem complicated and its connection with the vanishing of the determinant D, with elements given by (4.15), rather obscure. However, in order to apply the rule it is not necessary to use D explicitly and the number of terms obtained is usually quite small. As a first example consider the supercharacter $\{\bar{1}^2; 2^3\}$ of $\mathrm{U}(2/1)$. Spelt out in full (4.24) in this case takes the form

$$\begin{vmatrix} \{\bar{0}\}+\bar{x}_1\varepsilon\{\bar{0}\} & \{\bar{1}^3\}+\bar{x}_2\varepsilon\{\bar{1}\} & \{1\}+x_1\bar{\varepsilon}\{1\} & \{0\} \\ - & \{\bar{1}^2\}+\bar{x}_2\varepsilon\{\bar{0}\} & \{1^2\}+x_1\bar{\varepsilon}\{1^2\} & \{1\} \\ - & \{\bar{1}\} & \{1^3\}+x_1\bar{\varepsilon}\{1^3\} & \{1^2\}+x_2\bar{\varepsilon}\{0\} \\ - & \{\bar{0}\} & \{1^4\}+x_1\bar{\varepsilon}\{1^4\} & \{1^3\}+x_2\bar{\varepsilon}\{1\} \end{vmatrix} = 0. \tag{5.1}$$

The constant term, ultimately independent of both x_1 and x_2, can be written in the form

$$(1+x_1\bar{x}_1)\begin{vmatrix} \{\bar{0}\} & \{\bar{1}^3\} & \{1\} & \{0\} \\ - & \{\bar{1}^2\} & \{1^2\} & \{1\} \\ - & \{\bar{1}\} & \{1^3\} & \{1^2\} \\ - & \{\bar{0}\} & \{1^4\} & \{1^3\} \end{vmatrix} + (1+x_1\bar{x}_1)x_2\bar{x}_2\begin{vmatrix} \{\bar{0}\} & \{\bar{1}\} & \{1\} & - \\ - & \{\bar{0}\} & \{1^2\} & - \\ - & - & \{1^3\} & \{0\} \\ - & - & \{1^4\} & \{1\} \end{vmatrix} = 0 \tag{5.2}$$

where the dependence on the factors $x_i\bar{x}_i = 1$ has been displayed in order to make the origin of the various contributions transparent. It should be noted that the connection between (4.25) and (2.4) is made by means of (4.28) which links (4.26) and (2.6). In the example under consideration

$$h_1 = h_3 = 0 = h_{12} \qquad h_2 = h_4 = 2 = h_{21}. \tag{5.3}$$

Although the condition $h_{12} = 0$ is very important in attesting to the fact that in this case the supercharacter is indeed non-standard, the corresponding contributions to the modification rule (2.4) may be neglected in that they contribute an equal weighting to every distinct term. This is exemplified by the common factor $(1+x_1\bar{x}_1)$ appearing in (5.2). The removal of the continuous boundary strips of length $h_{21} = 2$ is exhibited in figure 4 where the diagrams are in one-to-one correspondence with the terms of (5.2). The final result is thus the $\mathrm{U}(2/1)$ modification rule

$$\{\bar{1}^2; 2^3\} = -\{21^2\}. \tag{5.4}$$

This illustrates another important point, namely that the symbols $\{\bar{\tau}; \sigma\}$ with which we have been dealing are to be thought of as denoting not irreducible representations of $\mathrm{U}(M/N)$ but merely supercharacters of $\mathrm{U}(M/N)$, defined as supersymmetric functions in terms of $\mathrm{U}(M)$ and $\mathrm{U}(N)$ characters via (3.1).

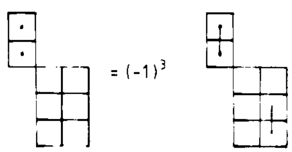


Figure 4.

As a second example consider the U(2/1) supercharacter $\{\overline{2^21}; 3^22\}$. This time (4.24) yields

$$\begin{vmatrix} \{\bar{1}^2\}+\bar{x}_1\varepsilon\{\bar{0}\} & \{\bar{1}^4\}+\bar{x}_2\varepsilon\{\bar{1}\} & \{1\} & \{0\} & - \\ \{\bar{1}\} & \{\bar{1}^3\}+\bar{x}_2\varepsilon\{\bar{0}\} & \{1^2\}+x_1\bar{\varepsilon}\{0\} & \{1\} & - \\ \{\bar{0}\} & \{\bar{1}^2\} & \{1^3\}+x_1\bar{\varepsilon}\{1\} & \{1^2\} & \{0\} \\ - & \{\bar{1}\} & \{1^4\}+x_1\bar{\varepsilon}\{1^2\} & \{1^3\}+x_2\bar{\varepsilon}\{0\} & \{1\} \\ - & \{\bar{0}\} & \{1^5\}+x_1\bar{\varepsilon}\{1^3\} & \{1^4\}+x_2\bar{\varepsilon}\{1\} & \{1^2\} \end{vmatrix} = 0. \tag{5.5}$$

Setting the constant term to zero gives

$$\begin{vmatrix} \{\bar{1}^2\} & \{\bar{1}^4\} & \{1\} & \{0\} & - \\ \{\bar{1}\} & \{\bar{1}^3\} & \{1^2\} & \{1\} & - \\ \{\bar{0}\} & \{\bar{1}^2\} & \{1^3\} & \{1^2\} & \{0\} \\ - & \{\bar{1}\} & \{1^4\} & \{1^3\} & \{1\} \\ - & \{\bar{0}\} & \{1^5\} & \{1^4\} & \{1^2\} \end{vmatrix} + x_1\bar{x}_1 \begin{vmatrix} \{\bar{0}\} & \{\bar{1}^4\} & - & \{0\} & - \\ - & \{\bar{1}^3\} & \{0\} & \{1\} & - \\ - & \{\bar{1}^2\} & \{1\} & \{1^2\} & \{0\} \\ - & \{\bar{1}\} & \{1^2\} & \{1^3\} & \{1\} \\ - & \{\bar{0}\} & \{1^3\} & \{1^4\} & \{1^2\} \end{vmatrix}$$

$$+x_2\bar{x}_2 \begin{vmatrix} \{\bar{1}^2\} & \{\bar{1}\} & \{1\} & - & - \\ \{\bar{1}\} & \{\bar{0}\} & \{1^2\} & - & - \\ \{\bar{0}\} & - & \{1^3\} & - & \{0\} \\ - & - & \{1^4\} & \{0\} & \{1\} \\ - & - & \{1^5\} & \{1\} & \{1^2\} \end{vmatrix} + x_1\bar{x}_1x_2\bar{x}_2 \begin{vmatrix} \{\bar{0}\} & \{\bar{1}\} & - & - & - \\ - & \{\bar{0}\} & \{0\} & - & - \\ - & - & \{1\} & - & \{0\} \\ - & - & \{1^2\} & \{0\} & \{1\} \\ - & - & \{1^3\} & \{1\} & \{1^2\} \end{vmatrix} = 0. \tag{5.6}$$

In this example

$$h_1 = h_3 = 2 = h_{12} \qquad h_2 = h_4 = 3 = h_{21}. \tag{5.7}$$

Clearly certain columns have to be interchanged in the determinants of (5.6) in order to express them in the regular manner of (3.8). This is all taken care of by the strip removal procedure, as shown in figure 5. Hence in U(2/1)

$$\{\overline{2^21}; 3^22\} = \{\bar{1}^3; 3^2\} - \{\bar{2}; 31^2\} + \{3\}. \tag{5.8}$$

This example indicates, as has been emphasised, that the relationship between supercharacters associated with composite Young diagrams and irreducible representations is by no means trivial. Any notion that the relationship might be one-to-one is plainly seen to be mistaken. This difficulty is connected of course to the existence of both typical and atypical irreducible representations (Kac 1978). Just as in the case of OSp(M/N) the distinction between typical and atypical supercharacters of U(M/N) can be extended from the standard to the non-standard case. The modification rules (5.4) and (5.8) are examples appropriate to non-standard supercharacters which are typical and atypical, respectively. The distinguishing feature is the number of standard supercharacters appearing in the result. If this number is one

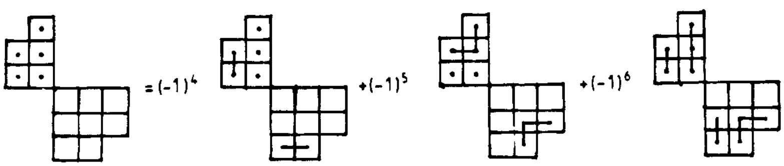


Figure 5.

then the supercharacter is typical and corresponds to a single typical irreducible representation of $U(M/N)$. If it is more than one then the supercharacters are all atypical and the correspondence with irreducible representations is more complicated.

The justification for these remarks lies outside the scope of the present paper but will be the subject of a further article in which the precise connection between the supercharacters defined here and those of Kac (1978) is discussed in detail.

The discerning reader will have noticed a degree of freedom in the derivation of identities relating supercharacters to one another. The first degree of freedom lies in the selection of the constant term independent of all x_i in passing from (4.24) to (4.25). It would have been possible to pick out, for example, from (5.5) not the constant terms but the terms linear in $\bar{x}_1$ and ultimately independent of x_2, giving

$$\bar{x}_1\varepsilon \begin{vmatrix} \{\bar{0}\} & \{\bar{1}^4\} & \{1\} & \{0\} & - \\ - & \{\bar{1}^3\} & \{1^2\} & \{1\} & - \\ - & \{\bar{1}^2\} & \{1^3\} & \{1^2\} & \{0\} \\ - & \{\bar{1}\} & \{1^4\} & \{1^3\} & \{1\} \\ - & \{\bar{0}\} & \{1^5\} & \{1^4\} & \{1^2\} \end{vmatrix} + \bar{x}_1 x_2 \bar{x}_2 \varepsilon \begin{vmatrix} \{\bar{0}\} & \{\bar{1}\} & \{1\} & - & - \\ - & \{\bar{0}\} & \{1^2\} & - & - \\ - & - & \{1^3\} & - & \{0\} \\ - & - & \{1^4\} & \{0\} & \{1\} \\ - & - & \{1^5\} & \{1\} & \{1^2\} \end{vmatrix} = 0. \tag{5.9}$$

Rearranging columns this gives the $U(2/1)$ modification rule

$$\{\bar{1}^3; 3^2 2\} = \{31^2\}$$

which could however have been obtained equally well from (4.25) and correspondingly (2.4) through the correct choice of D for the supercharacter $\{\bar{1}^3; 3^2 2\}$. Thus the exploitation of the first degree of freedom in the derivation of the modification rule does not lead to any new results.

However a second degree of freedom lies in the selection of the rightmost $N+1$ columns of the contravariant portion of det D and the leftmost $N+1$ columns of the covariant portion. It is clear that as long as precisely $N+1$ pairs of columns are selected, one member of each pair from each portion, then an $(m-N)\times(N+1)$ block of zeros can be obtained by multiplication of D by M and appropriate column manipulations. This then leads to the identity det $D=0$ and hence to alternative modification rules and of course alternative definitions of standardness. However these definitions are not all equivalent in that a supercharacter which is standard in one scheme may well be non-standard and therefore subject to further modification in another. Nonetheless the choice of columns made here and the definition (2.2) of standardness may be shown to be the best possible (Cummins 1986) in that this is the only scheme for which all standard supercharacters remain standard in every alternative scheme. There is one important proviso and that is that these schemes are restricted to those for which the resulting modification rule has no dependence on the $U(M/N)$ superdeterminant supercharacter ε. That is to say that the modification rules apply strictly to that class of supercharacters associated with Young diagrams. Therefore with this proviso the ideal modification rule has been derived, namely (2.4). All non-standard supercharacters specified by Young diagrams, whether composite or not, may be expressed in terms of standard supercharacters by means of (2.4), iterating if necessary.

References

Abramsky Y J and King R C 1970 *Nuovo Cimento* A **67** 153-216
Balantekin A B and Bars I 1981a *J. Math. Phys.* **22** 1149-62

—— 1981b *J. Math. Phys.* **22** 1810-8
Bars I 1984 *Introduction to Supersymmetry in Particle and Nuclear Physics* ed O Castanos, H Frank and L Urritia (New York: Plenum) pp 107-84
Bars I, Morel B and Ruegg H 1983 *J. Math. Phys.* **24** 2253-62
Black G R E, King R C and Wybourne B G 1983 *J. Phys. A: Math. Gen.* **16** 1555-9
Cummins C J 1986 *PhD Thesis* University of Southampton
Cummins C J and King R C 1987 *J. Phys. A: Math. Gen.* **20** 3103-20
Delduc F and Gourdin M 1984 *J. Math. Phys.* **25** 1651-61
—— 1985 *J. Math. Phys.* **26** 1865-79
Dondi P H and Jarvis P D 1981 *J. Phys. A: Math. Gen.* **14** 547-63
Kac V G 1978 *Lecture Notes in Mathematics* vol 676 (Berlin: Springer) pp 597-626
King R C 1970 *J. Math. Phys.* **11** 280-94
—— 1971 *J. Math. Phys.* **12** 1588-98
—— 1983a *Lecture Notes in Physics* vol 180 (Berlin: Springer) pp 41-4
—— 1983b *Ars. Comb.* **16A** 269-87
—— 1986 *Proc. 14th ICGTMP* ed Y M Cho (Singapore: World Scientific) pp 199-204
Littlewood D E 1940 *The Theory of Group Characters* (Oxford: Clarendon)